an upright position, and at a given angle of inclination θ_1, let ρ_0 and ρ_1 be those radii respectively; then make

$$r=\frac{2(\rho_1-\rho_0)}{\theta_1^2}. \qquad (2)$$

This will be the radius of the required circle; and its positive or negative sign will show whether it is to be laid off downwards or upwards from the metacentre. For any given angle of inclination the radius of curvature of the metacentric involute will be given by equation (1), which may also be put in the following form:

$$\rho=\rho_0+(\rho_1-\rho_0)\frac{\theta^2}{\theta_1^2}. \qquad (3)$$

Let δ be the depth of the ship's centre of gravity below her metacentre, and p the perpendicular let fall from that centre of gravity upon the radius of curvature of the metacentric involute at any given angle of inclination θ; then

$$p=(\delta-r)\sin\theta+r\theta; \qquad (4)$$

and the moment of stability is

$$p\times\text{displacement}. \qquad (5)$$

It is obvious that the condition of *isochronous rolling* is that $\delta-r=0$; that is to say, that the centre of the circle which is the evolute of the metacentric evolute shall coincide with the ship's centre of gravity; a proposition already demonstrated by me in a paper read to the Institution of Naval Architects in 1864, and published in their Transactions, vol. v. p. 35.

[*Postscript.*—Received March 11, 1867.]

Since the above was written, I have been informed by Mr. Merrifield, to whom I had communicated my proposed modification of his method, that it has been tried at the Royal School of Naval Architecture and found to answer well.

II. "On the Theory of the Maintenance of Electric Currents by Mechanical Work without the use of Permanent Magnets." By J. Clerk Maxwell, F.R.S. Received February 28, 1867.

The machines lately brought before the Royal Society by Mr. Siemens and Professor Wheatstone consist essentially of a fixed and a moveable electromagnet, the coils of which are put in connexion by means of a commutator.

The electromagnets in the actual machines have cores of soft iron, which greatly increase the magnetic effects due to the coils; but, in order to simplify the expression of the theory as much as possible, I shall begin by

supposing the coils to have no cores, and, to fix our ideas, we may suppose them in the form of rings, the smaller revolving within the larger on a common diameter.

The equations of the currents in two neighbouring circuits are given in my paper "On the Electromagnetic Field"*, and are there numbered (4) and (5),

$$\xi = \mathrm{R}x + \frac{d}{dt}(\mathrm{L}x + \mathrm{M}y),$$

$$\eta = \mathrm{S}y + \frac{d}{dt}(\mathrm{M}x + \mathrm{N}y),$$

where x and y are the currents, ξ and η the electromotive forces, and R and S the resistances in the two circuits respectively. L and N are the coefficients of self-induction of the two circuits, that is, their potentials on themselves when the current is unity, and M is their coefficient of mutual induction, which depends on their relative position. In the electromagnetic system of measurement, L, M, and N are of the nature of lines, and R and S are velocities. L may be metaphorically called the "electric inertia" of the first circuit, N that of the second, and L+2M+N that of the combined circuit.

Let us first take the case of the two circuits thrown into one, and the two coils relatively at rest, so that M is constant. Then

$$(\mathrm{R}+\mathrm{S})x + \frac{d}{dt}(\mathrm{L}+2\mathrm{M}+\mathrm{N})x = 0, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

whence

$$x = x_0 e^{-\frac{\mathrm{R}+\mathrm{S}}{\mathrm{L}+2\mathrm{M}+\mathrm{N}}t}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where x_0 is the initial value of the current. This expression shows that the current, if left to itself in a closed circuit, will gradually decay.

If we put

$$\frac{\mathrm{L}+2\mathrm{M}+\mathrm{N}}{\mathrm{R}+\mathrm{S}} = \tau, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

then

$$x = x_0 e^{-\frac{t}{\tau}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The value of the time τ depends on the nature of the coils. In coils of similar outward form, τ varies as the square of the linear dimension, and inversely as the resistance of unit of length of a wire whose section is the sum of the sections of the wires passing through unit of section of the coil.

In the large experimental coil used in determining the B.A. unit of resistance in 1864, τ was about ·01 second. In the coils of electromagnets τ is much greater, and when an iron core is inserted there is a still greater increase.

* Phil. Trans. 1865, p. 469.

Let us next ascertain the effect of a sudden change of position in the secondary coil, which alters the value of M from M_1 to M_2 in a time t_2-t_1, during which the current changes from x_1 to x_2. Integrating equation (1) with respect to t, we get

$$(R+S)\int_{t_1}^{t_2} x\,dt+(L+2M_2+N)x_2-(L+2M_1+N)x_1=0. \quad . \quad (5)$$

If we suppose the time so short that we may neglect the first term in comparison with the others, we find, as the effect of a *sudden* change of position,

$$(L+2M_2+N)x_2=(L+2M_1+N)x_1. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

This equation may be interpreted in the language of the dynamical theory, by saying that the electromagnetic momentum of the circuit remains the same after a sudden change of position. To ascertain the effect of the commutator, let us suppose that, at a given instant, currents x and y exist in the two coils, that the two coils are then made into one circuit, and that x' is the current in the circuit the instant after completion; then the same equation (1) gives

$$(L+2M+N)x'=(L+M)x+(N+M)y. \quad . \quad . \quad . \quad . \quad (7)$$

This equation shows that the electromagnetic momentum of the completed circuit is equal to the sum of the electromagnetic momenta of the separate coils just before completion.

The commutator may belong to one of four different varieties, according to the order in which the contacts are made and broken. If A, B be the ends of the first coil, and C, D those of the second, and if we enclose in brackets the parts in electric connexion, we may express the four varieties as in the following Table :—

(1)	(2)	(3)	(4)
(AC) (BD)	(AC) (BD)	(AC) (BD)	(AC) (BD)
(ABCD)	(ABC) (D)	(A) (BCD)	(A) (B) (C) (D)
(AD) (BC)	(ABCD)	(ABCD)	(AD) (BC)
	(AD) (BC)	(AD) (BC)	

In the first kind the circuit of both coils remains uninterrupted; and when the operation is complete, two equal currents in opposite directions are combined into one. In this case, therefore, $y=-x$, and

$$(L+2M+N)x'=(L-N)x. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

When there are iron cores in the coils, or metallic circuits in which independent currents can be excited, the electrical equations are much more complicated, and contain as many independent variables as there can be independent electromagnetic quantities. I shall therefore, for the sake of preserving simplicity, avoid the consideration of the iron cores, except in so far as they simply increase the values of L, M, and N.

I shall also suppose that the secondary coil is at first in a position in

which $M=0$, and that it turns into a position in which $M=-M$, which will increase the current in the ratio of $L+N$ to $L-2M+N$.

The commutator is then reversed. This will diminish the current in a ratio depending on the kind of commutator.

The secondary coil is then moved so that M changes from M to 0, which will increase the current in the ratio of $L+2M+N$ to $L+N$.

During the whole motion the current has also been decaying at a rate which varies according to the value of $L+2M+N$; but since M varies from $+M$ to $-M$, we may, in a rough theory, suppose that in the expression for the decay of the current $M=0$.

If the secondary coil makes a semirevolution in time T, then the ratio of the current x_1, after a semirevolution to the current x_0 before the semirevolution, will be

$$\frac{x_1}{x_0}=e^{\frac{T}{\tau}}\,r,$$

where

$$\tau=\frac{L+N}{R+S}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and r is a ratio depending on the kind of commutator.

For the first kind,

$$r=\frac{L-N}{L-2M+N}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

By increasing the speed, T may be indefinitely diminished, so that the question of the maintenance of the current depends ultimately on whether r is greater or less than unity. When r is greater than 1 or less than -1, the current may be maintained by giving a sufficient speed to the machine; it will be always in one direction in the first case, and it will be a reciprocating current in the second.

When r lies between $+1$ and -1, no current can be maintained.

Let there be p windings of wire in the first coil and q windings in the second, then we may write

$$L=lp^2, \quad M=mpq, \quad N=nq^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where l, m, n are quantities depending on the shape and relative position of the coils. Since $L-2M+N$ must always be a positive quantity, being the coefficient of self-induction of the whole circuit, $ln-m^2$, and therefore $LN-M$ must be positive. When the commutator is of the first kind, the ratio r is greater than unity, provided pm is greater than qn; and when

$$\frac{p}{q}=\frac{n}{m}\left(1+\sqrt{1-\frac{m^2}{ln}}\right),$$

$$r=\left(1-\frac{m^2}{ln}\right)^{-\frac{1}{2}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

which is the maximum value of r.

When the ratio of p to q lies between that of n to m and that of m to l, r lies between $+1$ and -1, and the current must decay; but when pl is less than qm, a reciprocating current may be kept up, and will increase most rapidly when

$$\frac{p}{q}=\frac{n}{m}\left(1-\sqrt{1-\frac{m^2}{ln}}\right),$$

and

$$r=-\left(1-\frac{m^2}{ln}\right)^{-\frac{1}{2}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

When the commutator is of the second kind, the first step is to close both circuits, so as to render the currents in them independent. The second circuit is then broken, and the current in it is thus stopped. This produces an effect on the first circuit by induction determined by the equation

$$Lx+My=Lx'+My'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

In this case $M=-M_0$, $y=x$, and $y'=0$, so that

$$(L-M)x=Lx'; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where x is the original, and x' the new value of the current.

The next step is to throw the circuits into one, M being now positive. If x'' be the current after this operation,

$$(L+M)x'=(L+2M+N)x''. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The whole effect of this commutator is therefore to multiply the current by the ratio

$$\frac{L^2-M^2}{L(L+2M+N)}.$$

The whole effect of the semirotation is to multiply the current by the ratio

$$\frac{L+2M+N}{L-2M+N}.$$

The total effect of a semirevolution supposed instantaneous is to multiply the current by the ratio

$$r=\frac{L^2-M^2}{L(L-2M+N)}.$$

If p and q be the number of windings in the first and second coils respectively, the ratio r becomes

$$r=\frac{l^2p^2-m^2q^2}{l(lp^2-2mpq+nq^2)};$$

which is greater than 1, provided $2lmp$ is greater than $(ln+m^2)q$. When

$$\frac{p}{q}=\frac{1}{2}\left(\frac{n}{m}+\frac{m}{l}\right)+\frac{1}{2}\sqrt{\frac{n^2}{m}+2\frac{n}{l}-3\frac{m^2}{l^2}},$$

we have for the maximum value of r,

$$r = 1 + \frac{2\frac{m}{l}}{\sqrt{\frac{n^2}{m^2} + 2\frac{n}{l} - 3\frac{m^2}{l^2} + \frac{n}{m} - \frac{m}{l}}}.$$

In the experiment of Professor Wheatstone, in which the ends of the primary coil were put in permanent connexion by a short wire, the equations are more complicated, as we have three currents instead of two to consider. The equations are

$$\mathrm{R}x + \frac{d}{dt}(\mathrm{L}x + \mathrm{M}y) = \mathrm{S}y + \frac{d}{dt}(\mathrm{M}x + \mathrm{N}y) = \mathrm{Q}z + \frac{d}{dt}\mathrm{K}z, \quad (17)$$

$$x + y + z = 0. \quad (18)$$

where Q, K, and z are the resistance, self-induction, and current in the short wire. The resultant equations are of the second degree; but as they are only true when the magnetism of the cores is considered rigidly connected with the currents in the coils, an elaborate discussion of them would be out of place in what professes to be only a rough explanation of the theory of the experiments.

Such a rough explanation appears to me to be as follows:—

Without the shunt, the current in the secondary coil is always in rigid connexion with that in the primary coil, except when the commutator is changing. With the shunt, the two currents are in some degree independent; and the secondary coil, whose electric inertia is small compared with that of the primary, can have its current reversed and varied without being clogged by the sluggish primary coil.

On the other hand, the primary coil loses that part of the total current which passes through the shunt; but we know that an iron core, when highly magnetized, requires a great increase of current to increase its magnetism, whereas its magnetism can be maintained at a considerable value by a current much less powerful. In this way the diminution in resistance and self-induction due to the shunt may more than counterbalance the diminution of strength in the primary magnet.

Also, since the self-induction of the shunt is very small, all instantaneous currents will run through it rather than through the electromagnetic coils, and therefore it will receive more of the heating effect of variable currents than a comparison of the resistances alone would lead us to expect.